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ON THE GENERALIZATION OF PRANDTL'S SOLUTION IN SPHERICAL COORDINATES*

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The generalization of Prandtl's solution /1/ on compression of an ideal rigid-plastic layer by rough parallel plates is considered in a spherical coordinate system.

Extension of Prandtl's solution to the case of plane deformation appeared in /2,3/, and in /4/ a generalization of that solution was presented in cylindrical coordinates.

Consider the axisymmetric equilibrium state in a spherical coordinate system, assuming that

$$\sigma_{ij} = \sigma_{ij}(\rho, \theta), \quad w = 0, \quad \tau_{\rho\varphi} = \tau_{\theta\varphi} = \varepsilon_{\rho\varphi} = \varepsilon_{\theta\varphi} = 0$$

where w is the component of displacement rate along the φ axis.

The equations of equilibrium assume the form

$$\frac{\partial\sigma_\rho}{\partial\rho} + \frac{1}{\rho}\frac{\partial\tau_{\rho\theta}}{\partial\theta} + \frac{1}{\rho}[2\sigma_\rho - \sigma_\theta - \sigma_\varphi + \tau_{\rho\theta}\operatorname{ctg}\theta] = 0 \tag{1}$$
$$\frac{\partial\tau_{\rho\theta}}{\partial\rho} + \frac{1}{\rho}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{1}{\rho}[(\sigma_\theta - \sigma_\varphi)\operatorname{ctg}\theta + 3\tau_{\rho\theta}] = 0$$

In the case of axisymmetric state we use the plasticity condition that corresponds to the Tresca prism edge

$$(\sigma_\rho - \sigma_\theta)^2 + 4\tau_{\rho\theta}^2 = 4k^2, \quad \sigma_\varphi = {}^1/_2(\sigma_\rho + \sigma_\theta) + k \tag{2}$$

All stress components are subsequently normalized with respect to double the yield stress $2k$.

Assuming that

$$\tau_{\rho\theta} = \tau_{\rho\theta}(\theta) \tag{3}$$

from (2) and (3) we obtain

$$\sigma_\rho - \sigma_\theta = \sqrt{1 - \tau_{\rho\theta}(\theta)} \tag{4}$$

Using (2) – (4) we rewrite the equilibrium equations in the form

$$\frac{\partial\sigma_\rho}{\partial\rho} + \frac{N(\theta)}{\rho} = 0, \quad \frac{\partial\sigma_\theta}{\partial\theta} + M(\theta) = 0 \tag{5}$$
$$N(\theta) = \frac{d\tau_{\rho\theta}}{d\theta} + 3\sqrt{1 - \tau_{\rho\theta}^2(\theta)} + \tau_{\rho\theta}\operatorname{ctg}\theta - 1$$
$$M(\theta) = 3\tau_{\rho\theta} - \left(\sqrt{1 - \tau_{\rho\theta}^2(\theta)} + 1\right)\operatorname{ctg}\theta$$

Integration of Eqs.(5) yields

$$\sigma_\rho = N(\theta)\ln\rho + \chi(\theta), \quad \sigma_\theta = P(\theta) + \varkappa(\rho), \quad P(\theta) = \int M(\theta)\,d\theta \tag{6}$$

where $(\chi(\theta), \varkappa(\rho)$ are arbitrary functions.

From (6) and (4) we have

$$\sigma_\rho - \sigma_\theta = N(\theta)\ln\rho + \chi(\theta) - P(\theta) - \varkappa(\rho) = \sqrt{1 - \tau_{\rho\theta}^2(\theta)} \tag{7}$$

It is necessary to set

$$N(\theta) = C, \quad \varkappa(\rho) = C\ln\rho, \quad C = \text{const} \tag{8}$$

In conformity with (5) and (8) we obtain for the determination of $\tau_{\rho\theta}(\theta)$ the equation

$$\frac{d\tau_{\rho\theta}}{d\theta} + 3\sqrt{1 - \tau_{\rho\theta}^2} + \tau_{\rho\theta}\operatorname{ctg}\theta = C + 1 = A_1 \tag{9}$$

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Suppose that the solution of Eq.(9) has been determined. Then in conformity with (5)–(7) functions $M(\theta)$, $P(\theta)$ and function

$$\chi(\theta) = \sqrt{1-\tau_{\rho\theta}^2} + P(\theta) \tag{10}$$

are also determined. Thus, the stress state has been completely determined.

For the determination of deformation kinematics we use the conditions of isotropy and incompressibility

$$\frac{\varepsilon_\rho - \varepsilon_\theta}{\sigma_\rho - \sigma_\theta} = \frac{\varepsilon_{\rho\theta}}{\tau_{\rho\theta}}, \quad \varepsilon_\rho + \varepsilon_\theta + \varepsilon_\varphi = 0 \tag{11}$$

We seek the displacement components of the form

$$u = \mu(\rho) + \rho\nu(\theta), \quad v = \rho\gamma(\theta)$$

For components of the deformation rate we then obtain

$$\varepsilon_\rho = \frac{d\mu}{d\rho} + \nu, \quad \varepsilon_\theta = \frac{\mu}{\rho} + \nu + \frac{d\gamma}{d\theta}$$
$$\varepsilon_\varphi = \frac{\mu}{\rho} + \nu + \gamma \operatorname{ctg}\theta, \quad \varepsilon_{\rho\theta} = \frac{1}{2}\frac{d\nu}{d\theta} \tag{12}$$

From Eqs.(11) and (12) we have the equalities

$$\frac{d\mu}{d\rho} - \frac{\mu}{\rho} - \frac{d\gamma}{d\theta} = \frac{d\nu}{d\theta}\left(\frac{\sigma_\rho - \sigma_\theta}{2\tau_{\rho\theta}}\right)$$
$$\frac{d\mu}{d\rho} + \frac{2\mu}{\rho} + 3\nu + \frac{d\gamma}{d\theta} + \gamma \operatorname{ctg}\theta = 0 \tag{13}$$

from which follows

$$\frac{d\mu}{d\rho} - \frac{\mu}{\rho} = C_1, \quad \frac{d\mu}{d\rho} + \frac{2\mu}{\rho} = C_2; \quad C_1, C_2 = \text{const} \tag{14}$$

From formulas (14) we obtain

$$\mu = B\rho, \quad B = \frac{1}{3}(C_2 - C_1) = \text{const} \tag{15}$$

Using (15) we rewrite equalities (13) in the form

$$\frac{d\nu}{d\theta} + \frac{2\tau_{\rho\theta}}{\sigma_\rho - \sigma_\theta}\frac{d\gamma}{d\theta} = 0, \quad 3\nu + \frac{d\gamma}{d\theta} + \gamma \operatorname{ctg}\theta + 3B = 0 \tag{16}$$

Differentiating the second of equalities (16) and eliminating $d\nu/d\theta$, for the determination of function $\gamma(\theta)$ we obtain the equation

$$\frac{d^2\gamma}{d\theta^2} + \frac{d\gamma}{d\theta}\left[\operatorname{ctg}\theta - 6\frac{\tau_{\rho\theta}}{\sigma_\rho - \sigma_\theta}\right] - \frac{\gamma}{\sin^2\theta} = 0 \tag{17}$$

It is characteristic that a second order equation determined $\gamma(\theta)$. As in the case of solution in cylindrical coordinates /4/, there are two independent solutions for the components of displacements and deformations. The solution of Eq.(17) enables us to determine completely the components of displacement and deformation.

The analytic solution of Eq.(9) presents definite difficulties. Let us consider the method of approximate solution.

Let a layer of perfectly plastic material be compressed by two approaching tapered plates symmetric relative to the plane $\theta = \pi/2$ (Fig.1). We set $\alpha = \pi/2 - \theta$ and expand all components in series in powers of α, retaining smalls of up to the third order. We have

$$\tau_{\rho\theta} = \tau_1\alpha + \tau_2\alpha^2 + \tau_3\alpha^3 + \tau_4\alpha^4 + \ldots \tag{18}$$
$$(1 - \tau_{\rho\theta}^2)^{1/2} = 1 - {}^1/_2\,\tau_1^2\alpha^2 - \tau_1\tau_2\alpha^3 + \ldots, \quad \tau_{\rho\theta}\operatorname{tg}\alpha = \tau_1\alpha^2 + \tau_2\alpha^3 + \ldots$$

From (18) and (9), equating terms at like powers, we obtain

$$\tau_1 = 3 - A_1 = A, \ \tau_2 = 0, \ \tau_3 = A/3 - A^2/2, \ \tau_4 = 0 \tag{19}$$

Thus the expression for $\tau_{\rho\theta}$ is of the form

$$\tau_{\rho\theta} = A\alpha + (A/3 - A^2/2)\,\alpha^3 \tag{20}$$

The constant A is determined by conditions at the boundary $\alpha = \alpha^*$

$$\alpha^{*2} A = \left(1 + \frac{\alpha^{*2}}{3}\right) \mp \left[\left(1 + \frac{\alpha^{*2}}{3}\right)^2 - 2\alpha^*\right]^{1/2} \tag{21}$$

Then, using formulas (20) and (6), we can obtain the solution in terms of stresses.

We solve Eq.(17) using the approximate methods. We pass to the small angle $\alpha = \pi/2 - \theta$ and seek the solution in terms of expansion in powers of α

$$\gamma = \gamma_1\alpha + \gamma_2\alpha^2 + \gamma_3\alpha^3 + \ldots, \quad \nu = \nu_0 + \nu_1\alpha + \nu_2\alpha^2 + \nu_3\alpha^3 + \ldots \tag{22}$$

We substitute (22) into (15), equate coefficients at like powers of α, and express all unknowns in terms of ν_0. We obtain

$$\nu_1 = 0, \quad \gamma_1 = -3(B + \nu_0), \quad \nu_2 = -3\tau_1(B + \nu_0), \quad \gamma_2 = 0 \tag{23}$$

$$\nu_3 = 0, \quad \gamma_3 = (B + \nu_0)(\tau_1 + 1), \quad \nu_4 = \frac{B + \nu_0}{4}\left[6\tau_1(\tau_1 + 1) - \tau_1^3 + \frac{\tau_3}{3}\right], \quad \gamma_4 = 0$$

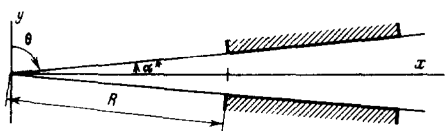

Fig.1

This solution corresponds to compression of a perfectly plastic material by rough bevelled plates with $\theta = \text{const}$.

Let us consider a particular case of the derived solution. We introduce new variables (so that $R\alpha^* = h$)

$$x = \rho - R, \quad y = R\alpha, \quad \sigma_x = \sigma_\rho, \quad \sigma_y = \sigma_\theta, \quad \tau_{xy} = \tau_{\rho\theta}$$

When passing to limit, as $\theta \to \pi/2$ or $\alpha \to 0$, $R \to \infty$, relations of the spherical deformation state obviously become relations of the plane deformation state in which the bevelled plate surfaces are parallel. In the first approximation the obtained solution becomes the linearized Prandtl solution.

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